COMP3161/9164 Homework Example

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1 M/L/N Example

Recall our language of matched parentheses

$$\{\epsilon, (), (()), ()(), \ldots\}$$

This is the definition of **M**:

$$\frac{}{\varepsilon\ \mathbf{M}}\mathsf{M_E} \qquad \frac{\mathsf{s}\ \mathbf{M}}{(\mathsf{s})\ \mathbf{M}}\mathsf{M_N} \qquad \frac{\mathsf{s_1}\ \mathbf{M} \quad \mathsf{s_2}\ \mathbf{M}}{\mathsf{s_1 s_2}\ \mathbf{M}}\mathsf{M_J}$$

This is the unambiguous definition of **L** and **N**:

$$\frac{}{\varepsilon\ \mathbf{L}}\mathsf{L_E} \qquad \frac{\mathsf{s}\ \mathbf{L}}{(\mathsf{s})\ \mathbf{N}}\mathsf{N_N} \qquad \frac{\mathsf{s_1}\ \mathbf{N} \quad \mathsf{s_2}\ \mathbf{L}}{\mathsf{s_1 s_2}\ \mathbf{L}}\mathsf{L_J}$$

This is the essential lemma 1 in the proof that **M** is contained in **L**:

$$\frac{\mathsf{s}\ \mathbf{L} \quad \mathsf{t}\ \mathbf{L}}{\mathsf{st}\ \mathbf{L}}$$

2 Homework

Prove Lemma 1

This is my solution.

Goal: We must show $\forall\ \mathsf{s}.\ \mathsf{s}\ \mathbf{L} \longrightarrow (\forall\ \mathsf{t}.\ \mathsf{t}\ \mathbf{L} \longrightarrow \mathsf{st}\ \mathbf{L})$

We will prove this by rule induction, proving $\mathsf{P(s)}$ for all s for which $\mathsf{s}\ \mathbf{L}$ is derivable and simultaneously proving $\mathsf{Q(s)}$ for all s for which $\mathsf{s}\ \mathbf{N}$ is derivable, where P and Q are defined by:

$$\mathsf{P(s)} \equiv ((\forall\ \mathsf{t}.\ \mathsf{t}\ \mathbf{L} \longrightarrow \mathsf{st}\ \mathbf{L}) \land \mathsf{s}\ \mathbf{L})$$

$$\mathsf{Q(s)} \equiv \mathsf{s}\ \mathbf{N}$$

We have three cases to show, corresponding to the three rules that define **L** and **N**.

Case 1: The base case associated with rule L_E.
We must show $P(\epsilon) \equiv \epsilon\ \mathbf{L}$.
This follows from L_E, as it is exactly the statement of L_E.

Case 2: The inductive case associated with L_J.
We assume two inductive hypotheses, $IH_1 \equiv Q(s_1) \equiv s_1\ \mathbf{N}$ and $IH_2 \equiv P(s_2)$.
We must show $P(s_1 s_2) \equiv (\forall\ t.\ t\ \mathbf{L} \longrightarrow s_1 s_2 t\ \mathbf{L}) \wedge s_1 s_2\ \mathbf{L}$.
Let us prove the two conjuncts separately.

Case 2 part 1: the left conjunct of case 2, and the tricky part of the proof.
We show this for all t by fixing some t and assuming that $t\ \mathbf{L}$. We must show $s_1 s_2 t\ \mathbf{L}$.
We will note (without further discussion) that juxtaposition is associative:

$$(s_1 s_2)t = s_1 s_2 t = s_1(s_2 t)$$

We have $IH_2 \equiv P(s_2) \equiv (\forall\ t.\ t\ \mathbf{L} \longrightarrow s_2 t\ \mathbf{L}) \wedge s_2\ \mathbf{L}$. We call its left conjunct IH_{2a} and its right conjunct IH_{2b}.
We show our goal by:

$$\dfrac{\dfrac{}{s_1\ \mathbf{N}}\,IH_1 \qquad \dfrac{\dfrac{}{t\ \mathbf{L}}\,\text{assumption}}{s_2 t\ \mathbf{L}}\,IH_{2a}}{s_1(s_2 t)\ \mathbf{L}}\,L_J$$

Case 2 part 2: the right conjunct of case 2, $s_1 s_2\ \mathbf{L}$.
This is shown simply by:

$$\dfrac{\dfrac{}{s_1\ \mathbf{N}}\,IH_1 \qquad \dfrac{}{s_2\ \mathbf{L}}\,IH_{2b}}{s_1 s_2\ \mathbf{L}}\,L_J$$

Case 3: The inductive case associated with N_N.
We assume the inductive hypothesis $IH \equiv P(s)$ and must show $Q(\ (s)\ ) \equiv (s)\ \mathbf{N}$.
The right conjunct of $P(s)$ is $s\ \mathbf{L}$, which we call IH_b. We show our goal simply by:

$$\dfrac{\dfrac{}{s\ \mathbf{L}}\,IH_b}{(s)\ \mathbf{N}}\,N_N$$